Tr. Inst. Mat., 1989, V. 13, P. 136-147.

ASYMPTOTTIC EXPANSIONS IN THE PROBLEM OF THE LENGTH OF THE LONGEST HEAD-RUN FOR MARKOV CHAIN WITH TWO STATES

S. YU. NOVAK

(abridged version)

Let $\{\xi_i, i\geq 0\}$ be a homogeneous Markov chain with states $\{0;1\}$, transition probabilities $p_{11}=\alpha$, $p_{00}=\beta$, $0<\alpha<1$, $\beta<1$, and initial distribution $\mathbb{P}(\xi_o=1)=p$. We set

$$\eta_n = \max\{k\leq n: \max_{o\leq i\leq n-k} 1\{\xi_{i+1}=\ldots=\xi_{i+k}=1\} = 1\} \tag{0}$$

Random variable η_n is known in literature as the length of the longest head-run.

V.L.Goncharov [1] proved that in the case of Bernoulli scheme we have: *for any* $j\in\mathbb{Z}$

$$\mathbb{P}(\eta_n-[\log n]<j) = \exp(-(1-\alpha)\alpha^{j-(\log n)}) + o(1) \quad (n\to\infty),$$

where $\log$ *is to base* $1/\alpha$, $[x]$ *is the integer part of* x, $(x)=x-[x]$.

Analogous results for more general situations were obtained in [2-7]. Assertions of LIL type were found in [4,5,8-13]. Moivre [14] seems to be the first who suggested to study the distribution of the length of the longest head-run.

The purpose of this article is to find asymptotic expansions in the limit theorem for the distribution of r.v. η_n.

§ 1. Main theorem.

Let $\gamma=(1-\alpha)(1-\beta)/\alpha(2-\alpha-\beta)$ and

$$Y_i(k,\phi) = \tag{1.1}$$

$$\phi^i \sum_{j=0}^{i} \sum_{d=0}^{j} T_{j-d} \sum_{\mu=0}^{d} Q_{\mu,d} \sum_{\nu=0}^{i-j} h(\nu,i-j)(\nu!)^{-1} \sum_{\lambda=0}^{\nu} C_\nu^\lambda (-1)^{d+\mu+\lambda} \cdot$$

$$\cdot \phi^{\mu+\lambda}((j+\mu)^{(\nu-\lambda)} - (d+\mu)(j+\mu-1)^{(\nu-\lambda)}\phi^{-1}) \qquad (i\geq 0)$$

where

$$\begin{aligned} i^{(d)} &\equiv i(i-1)\dots(i-d+1) \qquad (d\geq 1) \ , \\ i^{(0)} &= 1 \ , \quad i^{(-d)} = 0 \end{aligned} \tag{1.2}$$

functions T , Q , h are defined by formulae (2.7), (2.16), (2.12) .

Note that $Y_i(k,\phi)$, as a function of the first argument, is a polinomial of degree i ; it is a polinomial of degree $2i$ as a function of the second argument.

Theorem 1. *For any $m\geq 1$ there exists constant $C_m\equiv C(m,\alpha,\beta,\rho)$ such that for $n>C_m$ there holds*

$$\sup_{-\infty<j<+\infty} \left| \mathbb{P}(\eta_n-[\log n] < j) - e^{-\phi_{nj}} \sum_{i=0}^{m-1} n^{-i} Y_i(k_{nj},\phi_{nj}) \right|$$

$$\leq C_m(n^{-1}\ln n)^m \tag{1.3}$$

where $k_{nj}=j+[\log n]$, $\phi_{nj}=\gamma\alpha^{j-[\log n]}$.

Corollary. *For $n\to\infty$ we have*

$$\sup_{-\infty<j<+\infty} \left| \mathbb{P}(\eta_n-[\log n] < j) - \right.$$

$$- e^{-\phi_{nj}}(1+\phi_{nj}(1-\phi_{nj})n^{-1}\log n) \mid = O(n^{-1}) \tag{1.4}$$

Note that the first and the second terms of the expansion both do not depend on the initial distribution of the chain.

§ 2. Some auxiliary results

In the sequel letters C , c (with indexes or without) denote constants which depend on m and chain parameters only.

Teorem 2. *There exist constants* $q<1$ *and* $C<\infty$ *such that*

$$\sup_{k>C} \mid \mathbb{P}(\eta_n<k) - A(t_o)t_o^{-n-1} \mid \le Cq^n \tag{2.1}$$

where

$$A(t) = -V(t)/U'(t) \quad ,$$

$$V(t) \equiv V(t,k) = 1 - (\alpha+\beta-1)t -$$

$$- (p\alpha+(1-p)(1-\beta))\alpha^{k-1}t^k + p(\alpha+\beta-1)\alpha^k t^{k+1} \quad ,$$

$$U(t) \equiv U(t,k) = W(t) + (1-\alpha)(1-\beta)\alpha^{k-1}t^{k+1} \quad ,$$

$$W(t) = (1-t)(1-(\alpha+\beta-1)t) \quad ,$$

$t_o \equiv t_o(k)$ *is a root of* $U(t,k)$ *with minimal modulus.*

In the case of Bernoulli $B(\alpha)$ scheme we have $q=\alpha$ and $C=(2+\alpha(1+\alpha))/(1-\alpha)(1-\alpha^2)$.

Lemma 1. *For* $k\ge1$ *we have*

$$F(k,t) \equiv \sum_{n=o}^{\infty} \mathbb{P}(\eta_n<k)t^n = V(t)/U(t) \tag{2.2}$$

where $\eta_o \equiv 0$.

Denote $\varkappa = (\alpha+\beta-1)/(2-\alpha-\beta)$, $\delta = 1-p/\gamma-(1-p)/(1-\alpha)$, $\rho = (\alpha+\beta-1)/(1-\alpha)(1-\beta)$, $H_i \equiv 0 \quad (i<0)$,

$$H_i \equiv H_i(k) = \tag{2.6}$$

$$= 2^{-i} \sum_{j=o}^{[i/2]} C_{i+1}^{i-2j} (k+2\varkappa)^{i-2j} \left[(k+2\varkappa)^2 - 4(k-1)\varkappa \right]^j \qquad (i \geq 0)$$

We put

$$T_i \equiv T_i(k) = \sum_{j=o}^{3} q_j H_{i-j} \quad , \qquad (2.7)$$

where $q_o=1$, $q_1=\delta-\varkappa$, $q_2=\rho\alpha p-\varkappa\delta$, $q_3=-\varkappa\rho\alpha p$.

Lemma 2. For all k large enough we have

$$A(1+u) = \sum_{i=o}^{\infty} T_i u^i \qquad (2.8)$$

where $u \equiv u(k) = t_o(k)-1$.

Note that

$$| H_i(k) | \leq (k+2|\varkappa|)^i \quad , \quad | T_i(k) | \leq Ck^i \qquad (2.9)$$

We define polinomials $P_i(\cdot)$ by the equalities $P_o \equiv 0$,

$$P_m \equiv P_m(k) = \sum_{j=1}^{m} G_j(k) b_{m-j,j}(k) \qquad (k \geq m \geq 1) \quad ,$$

where $G_j(k) = \sum_{j=o}^{i} C_{k+1}^{i} \varkappa^{j-i}$ and

$$b_{l,j} \equiv b_{l,j}(k) = \sum_{i_1+\ldots+i_j=l} P_{i_1} \cdot \ldots \cdot P_{i_j} \qquad (l \geq 0 \ , \ j \geq 1)$$

Let $v \equiv v(k) = \gamma\alpha^k$.

Lemma 3. *For any* $m \geq 1$ *there exist constants* c_m , k_m *such that for* $k \geq k_m$ *we have*

$$\left| u/v - \sum_{i=o}^{m-1} P_i v^i \right| \leq c_m (kv)^m \qquad (2.10)$$

We introduce functions $\{\tilde{P}_i, i \geq 0\}$ by the equalities

$$\tilde{P}_i = P_i \qquad (0 \leq i \leq m) \quad ,$$

$$\tilde{P}_i v^m = u/v - \sum_{i=o}^{m-1} P_i v^i$$

We put also

$$b_{l,j,m} = \sum_{\substack{i_1+\dots+i_j=l \\ \max i_r \le m}} \tilde{P}_{i_1} \tilde{P}_{i_2} \dots \tilde{P}_{i_j}$$

Note that

$$b_{l,j} = \sum_{\nu=1}^{l} j^{(\nu)} h(\nu,l)/\nu! \qquad (l\ge 1, j\ge 1) \tag{2.11}$$

where $h(\nu,l) \equiv h(\nu,l,k)$ is a polinomial (as function of k) defined by the equalities $h(0,0) = 1$, $h(0,l) = 0$ $(l\ge 1)$,

$$h(\nu,l) \equiv h(\nu,l,k) = \sum_{1\le M\le \nu'} \sum_{(y,z)\in A(\nu,l,M)} (\nu!/z!) \cdot$$

$$\cdot \{P_{y_1}(k))^{z_1} \dots (P_{y_M}(k))^{z_M} \qquad (l\ge\nu\ge 1) \tag{2.12}$$

Here $\nu' = \min\{\nu; \sqrt{2l}\}$; $y = \{y_1,\dots,y_M\}$; $z = \{z_1,\dots,z_M\}$; $z! = z_1!\dots z_M!$;

$$A(\nu,l,M) = \left\{(y,z):\ 1\le y_1<\dots<y_M\ ;\ \min_i z_i \ge 1\ ;\ \sum_{i=1}^{M} z_i = \nu\ ;\ \sum_{i=1}^{M} y_i z_i = l\right\}$$

Similarly

$$b_{l,j,m} = \sum_{\nu\ge l/m}^{l} j^{(\nu)} h_m(\nu,l)/\nu! \quad , \tag{2.13}$$

where $h_m(0,0) = 1$, definition of $h_m(\nu,l)$ differs from that one of $h(\nu,l)$ by using $\tilde{P}_i$ instead of P_i and $A(\nu,l,M,m)$ instead of $A(\nu,l,M)$, where

$$A(\nu,l,M,m) = \left\{ (y,z)\in A(\nu.l,M)\ :\ \max_{1\le i\le M} y_i \le m \right\}$$

Note that $h_m(\nu,l) = h(\nu,l)$ as $l<m$ and

$$|h_m(\nu,l,k)| \le 2^m m^\nu (c_m k)^l \tag{2.14}$$

$$|b_{l,j,m}(k)| \le 2^m (m+1)^j (c_m k)^l \tag{2.15}$$

Lemma 4. *Let* $S_d(i) = \sum_{r=o}^{i} r^{(d)}$. *Then for* $d\geq 0$ *we have*

$$S_d(i) = (i+1)^{(d+1)}/(d+1) = i^{(d)} + i^{(d+1)}/(d+1)$$

Corollary.

$$S_d(i) = d \sum_{j=1}^{i} S_{d-1}(j-1) \qquad (d\geq 1)$$

$$(i+1)S_d(i-1) = (d+2)(d+1)^{-1} S_{d+1}(i) \qquad (i\geq 1)$$

$$S_d(i+1) = S_d(i) + dS_{d-1}(i) \qquad (d\geq 1)$$

Lemma 5. *Let coefficients* $r_j(i)$ *be defined by the equality*

$$(n+1)\circ\ldots\circ(n+i) \equiv \sum_{j=o}^{i} r_j(i) n^{i-j} \quad ,$$

and let

$$\tilde{Q}_{o,d} = 1 \ , \ \tilde{Q}_{j,d} = \sum_{1<l_1<l_1+1<\ldots l_j<d+j} (l_1 l_2 \ldots l_j)^{-1}$$

for $1\leq j<d$. *If* $d\geq 1$ *then we have*

$$r_d(i) = \sum_{j=o}^{d-1} \tilde{Q}_{j,d} \, S_{j+d}(i)$$

There follows from lemmas 4,5 that

$$r_d(i) = \sum_{j=o}^{d} Q_{j,d}(i+1)^{(j+d)} \qquad (d\geq 0) \tag{2.16}$$

where $Q_{o,o}=1$, $Q_{o,d}=0$ $(d\geq 1)$, $Q_{j,d} = (j+d)^{-1}\tilde{Q}_{j-1,d}$ $(1\leq j\leq d)$.

Lemma 6. *Let* $a,\nu\in\mathbb{Z}$; $\nu\geq 0$. *Then*

$$i^{(\nu)} = \sum_{\lambda=o}^{\nu} C_\nu^\lambda \, a^{(\nu-\lambda)}(i-a)^{(\lambda)} \tag{2.17}$$

§ 3. Proof of the main result.

We define $Y_{i,m} \equiv Y_{i,m}(k,\phi)$ by using $h_m(\nu,l)$ instead of $h(\nu,l)$ in formula (1.1) . In the sequel $\phi \equiv nv$.

Lemma 7. *For all k large enough we have*

$$A(l_o)l_o^{-n-1} = e^{-\phi} \sum_{i=o}^{\infty} n^{-i} y_{i,m} \tag{3.1}$$

Lemma 8. *Let $\psi = max(1;\phi)$. Then*

$$|y_{i,m}| \leq (c\psi^2 ln\ n)^i \tag{3.6}$$

Let $k(n) = log\ n - log\ ln\ n^m$ (log is to base $1/\alpha$) .

Lemma 9. *If $m>1$, then for all n large enough we have*

$$\sup_{k\in\mathbb{Z}} \left| \mathbb{P}(\eta_n<k) - e^{-\phi} \sum_{i=o}^{m-1} n^{-i} y_i \right| \leq Cq^n + \tag{3.8}$$

$$+ 2 \sup_{k\leq k(n)} e^{-\phi} \sum_{i=o}^{m-1} n^{-i} |y_i| + \sup_{k\geq k(n)} e^{-\phi} \sum_{i=m}^{\infty} n^{-i} |y_{i,m}| ,$$

where $q<1$.

Let

$$\tilde{\eta}_n = \max\{k\leq n: \max_{o\leq i\leq n-k} 1\{\xi_i=\ldots=\xi_{i+k-1}=1\} = 1\}$$

It is easy to see that assertion (1.3) holds if we define Y_i using $\tilde{T}_i$ instead of T_i , where $\tilde{T}_i = \sum_{j=o}^{3} \tilde{q}_j H_{i-j}$, $\tilde{q}_o=1$, $\tilde{q}_1 = 1 - \varkappa + \hat{\beta} - p/(1-\alpha)(1-\beta)$, $\tilde{q}_2=(1-\varkappa)\rho - \varkappa + \varkappa\hat{p}/(1-\alpha)(1-\beta)$, $\tilde{q}_3 = -\varkappa p$, $\hat{\beta} = \alpha(p+\beta-1)/(1-\alpha)(1-\beta)$.

§ 4. Remark on the rate of convergence.

§ 4. Remark on the rate of convergence.

Let $\{X_n n\geq 1\}$ be a Markov chain with state space $S = \{0,1,\dots,m\}$, transition probabilities p_{ij} and initial distribution $\bar{p}$. We define r.v. η_n by equality (0) , where $\zeta_i = 1\{X_i\in A\}$, $A = \{1,\dots,m\}$.

Let λ be a maximal eigenvalue of the matrix $U=\|p_{ij}\|_{ij\in A}$. We introduce r.v. ζ with distribution

$$\mathbb{P}(\zeta=1) = p_{oo} \quad , \quad \mathbb{P}(\zeta=i) = \bar{p}_{oA}U^{i-2}\bar{p}_{Ao} \quad (i\geq 2) \quad ,$$

where $\bar{p}_{oA} = \|p_{oj}\|_{j\in A}$, $\bar{p}_{Ao} = \|p_{io}\|_{i\in A}$. We suppose that there is only one class $\mathbb{C}$ of essential states, which has no cyclic subclasses; $A\cap\mathbb{C}\neq\emptyset$; $0<\lambda<1$; corresponding right eigenvector $\bar{z}$ of matrix U is positive: $z_j>0$ $(1\leq j\leq m)$.

Theorem 3. *Let* $\alpha(k)=\mathbb{P}(\zeta>k)$ *and*

$$\Delta(n,k) = | \mathbb{P}(\eta_n<k) - \exp(-n\alpha(k)/\mathbb{M}\zeta) | \tag{4.1}$$

Then $\sup\limits_{1\leq k\leq n} \Delta(n,k) = O(n\ (\ln n)\ )$ *as* $n\to\infty$.

Let τ_i be the i-th zero in the sequence $\{X_n n\geq 1\}$ and let $\zeta_i=\tau_i-\tau_{i-1}$. Then

$$\eta_n = \max\{ n-\tau_{\nu(n)} \ ; \max_{1\leq i\leq\nu(n)} \zeta_i-1 \} \tag{4.2}$$

where $\nu(n) = \max\{i: \tau_i\leq n \}$. The proof is based on the fact that $\mathbb{P}(\eta_n<k) \simeq \mathbb{M}(1-\alpha(k))^{\nu(n,k)}$, where $\nu(n,k) = \max\{r: \sum_{j=1}^{r}\zeta_j^{(k)}\leq n \}$, r.v.'s $\zeta_j^{(k)}$ are independent and have the distribution $\mathbb{P}(\zeta_j^{(k)}=i) = \mathbb{P}(\zeta_j=i\,|\,\zeta_j\leq k)$

References

1. Goncharov V.L. *From the domain of combinatorics* //Izv. Akad. Nauk SSSR, Ser.Mat.- 1944.- v.8,No 1.- 3-48.

2. Földes A. *The limit distribution of the length of the longest head-run* // rans. 8-th Prague Conf. Inform Th., Statist. Decis. Func., Rand Processes. - Prague, 1979. - 95-104.

3. Kusolitsch N. *Longest runs in Markov chains* // Probab. Statist. Inference. - 1982. - 223-230.

4. Novak S.Yu. *On the length of the longest head-run in Markov chains* // 4th Intern. Vilnius Conf. Probab. Th. Math. Statist. : Abstracts Commun. -Vilnius,1985. - v.2. - 267-268.

5. Novak S.Yu. *Time intervals of constant sojourn of a homogeneous Markov chain in a fixed subset of states* // Siberian Math. J. - 1988. - v.29, No. 1. - 100-109.

6. Mory T.F., Szekely G. *Asymptotic independence of "pure head" stopping times* // Statist. Probab. Letters. - 1984. - V.2,No 1. - 5-8.

7. Anisimov V.V., Cherniak A.I. *Limit theorems for certain rare functionals on Markov chains and semi-Markov processes* // Theor. Probab. Math. Statist. - 1983.-v.26. - 1-6.

8. Erdös P., Révész P. *On the length of the longest head-run* // Colloq. Math. Soc. J. Bolyai : Topics inform. theory. - 1975. - V. 16. - 219-228.

9. Samarova S.S. *On the number of time intervals that an ergodic Markov chain is continuously in a fixed state* // Dokl. Akad. Nauk SSSR. - 1981. - v.260, No 1. - 35-40.

10. Variakojis L. *On the maximum length of successful* series *in a enumerable Markov chain* // *Lietuvos Matem. Rinkinys* - 1986. - v.26, No 4. - 616-625.

11. Guibas L.J., Odlyzko A.M. *Long repetitive patterns in random sequences* // Z. Wahrscheinlichkeitstheor. Verw. Geb. - 1980. - B.53. - No 3. - 241-262.

12. Novak S. Yu. *On the length of the longest increasing run* // 1st Congr. Bernoulli Soc. Math. Statist. Probab. Th. - Tashkent, 1986. - v. 2. - 766.

13. Grill K. *Erdős-Révész type bounds for the length of the longest run from a stationary mixing sequence* // Probab. Th. Rel. Fields . - 1987. - v.75, No 1. - 77-85.

14. Moivre A. Doctrine of chances // L., 1738.

15. Novak S. Yu. *On the time that a homogeneous Markov chain is continuously in a finite subset of states* // Theor. Probab. Appl. - 1986. - v. 31, No 2. - 412-413.

16. Bitzadze A. V. *Theory of analitic functions of complex argument.* - Moscow: Nauka, 1984.

17. Feller W. *An introduction to probability theory and its applications.* - J. Wiley & Sons, 1970.

18. Borovkov A. A. *Probability Theory.* - Moscow: Nauka, 1986.

19. Borovkov A. A. *Mathematical Statistics.* - Moscow: Nauka, 1984.

Electrotechnical Institute
korp. 5 - 521
pr. Marksa 20
Novosibirsk 630092
Russia